5.6 Multistep Methods

We have tried two ideas to approx. IVP's for ODE's.

(I) Taylor Methods:

$$w_{i+1} = w_i + h\left[f(t_i,w_i) + \frac{h}{2} f'(t_i,w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i,w_i)\right]$$

Need to know derivatives of f at (t_i,w_i) previous point. or approximation. (t_i,w_i)

$$\tau_{i+1}(h) = O(h^n)$$

(II) R-K Methods

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h\left[a_1 f(t_i,w_i) + a_2 f\left(t_i+\alpha_2, w_i + \delta_2 f(t_i,w_i)\right)\right]$$

Need to know f at t_i or $t \in [t_i, t_{i+1}]$ and w_i

(III) Multistep Methods

: To compute w_{i+1} at t_{i+1}, the numerical method requires knowledge of (t_i,w_i), (t_{i-1},w_{i-1}), .., $(t_{i-(m-1)}, w_{i-(m-1)})$

and $f(t_i,w_i)$, .., $f((\;\;))$

Work examples 1st before definition

Def.– Consider $y' = f(t, y(t))$, $a \le t \le b$, $y(a) = \alpha$.

An m-step multistep method is defined by

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2, \ldots,$ $w_{m-1} = \alpha_{m-1}$ (Starting values.)

$$\boxed{\begin{aligned} w_{i+1} &= a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i-(m-1)} \\ &+ h\left[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \right. \\ &\left. + b_0 f(t_{i-(m-1)}, w_{i-(m-1)}) \right] \end{aligned}}$$

for $i = m-1, m, \ldots, N-1$, $h \equiv \dfrac{b-a}{N}$

Example 1.– ADAMS-BASHFORT 3-steps Explicit method.

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$

$$w_{i+1} = w_i + \frac{h}{12}\left[23 f(t_i, w_i) - 16 f(t_{i-1}, w_{i-1}) + 5 f(t_{i-2}, w_{i-2}) \right]$$

$i = 2, 3, \ldots, N-1$

It will be shown that $\tau_{i+1}(h) = O(h^3)$.

Example 2.– ADAMS-MOULTON 3-steps implicit method

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$

$$w_{i+1} = w_i + \frac{h}{24}\left[9 f(t_{i+1}, w_{i+1}) + 19 f(t_i, w_i) - 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2}) \right]$$

It will be shown $\tau_{i+1}(h) = O(h^4)$.

Derivation of ADAMS-BASHFORT m-step explicit method.

Consider IVP:

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha.$$

Integrating

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, \underset{\text{unknw.}}{y(t)})\,dt$$

The Integration is performed by approximating $f(t, y(t))$ in $[t_i, t_{i+1}]$ by an interpolating polyn. $P_{m-1}(t)$ of degree $m-1$.

$$\boxed{f(t, y(t)) = P_{m-1}(t) + \frac{f^{(m)}(\xi_i(t), y(\xi_i(t)))}{m!}(t-t_i)(t-t_{i-1})\ldots(t-t_{i-(m-1)})} \quad \xi_i(t) \in (t_{i-m+1}, t_i) \quad (3.1)$$

$P_{m-1}(t)$ is interpolating $f(t, y(t))$ at $(t_i, f_i), (t_{i-1}, f_{i-1}), \ldots, (t_{i-(m-1)}, f_{i-(m-1)})$

$f_i \equiv f(t_i, y(t_i))$

Then,

$$\boxed{y(t_{i+1}) \simeq y(t_i) + \int_{t_i}^{t_{i+1}} P_{m-1}(t)\,dt} \quad (3.2)$$

where $P_{m-1}(t) = a_0 + a_1(t-t_i) + a_2(t-t_i)(t-t_{i-1}) + \ldots + a_{m-1}(t - t_{i-(m-2)})$

m unknowns: $a_0, a_1, \ldots, a_{m-1}$
and m data points.

Application: Derivation of A-B 3-step explicit method

We need to construct $P_2(t)$ interpolating

$$\boxed{(t_i, f_i),\ (t_{i-1}, f_{i-1}),\ \text{and}\ (t_{i-2}, f_{i-2}).}$$

we use divided difference backward:

$t_i - t_{i+1} = h$

$$\begin{array}{lll}
t_{i-2} & f_{i-2} & \\
 & & \dfrac{f_{i-1}-f_{i-2}}{h} \longrightarrow \boxed{\dfrac{\dfrac{f_i - f_{i-1}}{h} - \dfrac{f_{i-1}-f_{i-2}}{h}}{2h}}\ (*) \\
t_{i-1} & f_{i-1} & \\
t_i & \boxed{f_i} & \boxed{\dfrac{f_i - f_{i-1}}{h}} = f[t_{i-1}, t_i] \qquad \to f[t_{i-2}, t_{i-1}, t_i]
\end{array}$$

$\hookrightarrow = f[t_i]$

then, $P_2(t) = f[t_i] + f[t_{i-1}, t_i](t - t_i) + f[t_{i-2}, t_{i-1}, t_i] * (t-t_i)(t-t_{i-1})$

$$(*) = \frac{f_i - 2f_{i-1} + f_{i-2}}{2h^2}$$

therefore,
$$\boxed{P_2(t) = f_i + \frac{1}{h}(f_i - f_{i-1})(t - t_i) + \frac{1}{2h^2}(f_i - 2f_{i-1} + f_{i-2}) * (t-t_i)(t-t_{i-1})}$$

It's easier to integrate introducing the new variable "s" $t \overset{(s)}{=} t_i + sh \Rightarrow dt = s\,dh$, $0 < s < 1$

$t(0) = t_i$, $t(1) = t_{i+1}$

then, $P_2(t) = P_2(t(s)) = P_2(t_i + sh) = \hat{P}_2(s)$

$$\boxed{\int_{t_i}^{t_{i+1}} P_2(t)\,dt = \int_0^1 \hat{P}_2(s)\,\underbrace{h\,ds}_{dt} = h\int_0^1 \hat{P}_2(s)\,ds}\quad (**)$$

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Notice, $t-t_i = t_i+sh-t_i = sh$, $(t-t_{i-1}) = t_i+sh-t_{i-1}$
$= t_i-t_{i-1}+sh$
$= h+sh.$

$$\hat{P}_2(s) = P_2(t_i+sh) = f_i + \frac{1}{\cancel{h}}(f_i - f_{i-1})s\cancel{h} +$$

$$+ \frac{1}{2\cancel{h^2}}(f_i - 2f_{i-1} + f_{i-2})\underbrace{sh(sh+h)}_{s(s+1)\cancel{h^2}} =$$

$$= \left(1+s+\frac{s^2+s}{2}\right)f_i + (-s-s^2-s)f_{i-1}$$

$$+\left(\frac{s^2+s}{2}\right)f_{i-2}$$

thus,

$$\frac{1}{h}\int_{t_i}^{t_{i+1}} P_2(t)dt = \int_0^1 \hat{P}_2(s)\,ds = \left[\int_0^1 \left(1+\frac{3}{2}s+\frac{s^2}{2}\right)ds\right]f_i - \left[\int_0^1 (s^2+2s)\,ds\right]f_{i-1}$$

$$+\left[\int_0^1 \left(\frac{s^2+s}{2}\right)ds\right]f_{i-2} =$$

$$= \left(s+\frac{3}{4}s^2+\frac{s^3}{6}\right)\Big|_0^1 f_i - \left(\frac{s^3}{3}+s^2\right)\Big|_0^1 f_{i-1} +$$

$$+\frac{1}{2}\left(\frac{s^3}{3}+\frac{s^2}{2}\right)\Big|_0^1 f_{i-2} =$$

$$= \left(1+\frac{3}{4}+\frac{1}{6}\right)f_i - \left(\frac{1}{3}+1\right)f_{i-1} + \left(\frac{1}{2}\right)\left(\frac{1}{3}+\frac{1}{2}\right)f_{i-2} =$$

$$= \left(\frac{12+9+2}{12}\right)f_i - \left(\frac{4}{3}\right)f_{i-1} + \frac{5}{12}f_{i-2}$$

$$= \frac{23}{12}f_i - \frac{4}{3}f_{i-1} + \frac{5}{12}f_{i-2}$$

Now,

$$y(t_{i+1}) \simeq y(t_i) + \int_{t_i}^{t_{i+1}} P_2(t)\,dt$$

$$\overset{(**)}{=} y(t_i) + h\int_0^1 \hat{P}_2(s)\,ds =$$

$$= y(t_i) + h\left[\frac{23}{12} f_i - \frac{4}{3} f_{i-1} + \frac{5}{12} f_{i-2}\right]$$

or
$$y(t_{i+1}) \simeq y(t_i) + \frac{h}{12}\left[23 f_i - 16 f_{i-1} + 5 f_{i-2}\right].$$

From where we obtain the 3-step A-B method

$$\boxed{\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2 \\ &w_{i+1} = w_i + \frac{h}{12}\left[23 f(t_i, w_i) - 16 f(t_{i-1}, w_{i-1}) + 5 f(t_{i-2}, w_{i-2})\right] \\ &\qquad i = 2, 3, \ldots, N-1 \end{aligned}} \qquad (6.1)$$

Other multistep methods can be derived in similar way.

<u>Def. and Deriv. of LTE.</u>

$\xi_i(t) \in (t_{i-2}, t_i)$

$$f(t, y(t)) = P_2(t) + \frac{f^{(3)}(\xi_i(t), y(\xi_i(t)))}{3!}(t-t_i)(t-t_{i-1})(t-t_{i-2})$$

Then,

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt = \int_{t_i}^{t_{i+1}} P_2(t)\,dt + \int_{t_i}^{t_{i+1}} \frac{f^{(3)}(\xi_i(t), y(\xi_i(t)))}{3!}(t-t_i)(t-t_{i-1})(t-t_{i-2})\,dt$$

Then,

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt = \frac{h}{2}\left[23 f_i - 16 f_{i-1} + 5 f_{i-2}\right]$$

$$\underbrace{+ \int_{t_i}^{t_{i+1}} \frac{f^{(3)}(\xi(t), y(\xi(t)))}{3!} (t-t_i)(t-t_{i-1})(t-t_{i-2})\,dt}_{R_i}$$

$t = t_i + sh$, $dt = h\,ds$

Now, $$R_i = \int_0^1 \frac{f^{(3)}(\xi(t(s)), y(\xi(t(s))))}{3!} (sh)(sh+h)(sh+2h)\,h\,ds$$

$$= \frac{h^4}{3!} \int_0^1 f^{(3)}(\xi(t(s)), y(\xi(t(s))))\, s(s+1)(s+2)\,ds$$

Weighted MVT

$$= \frac{h^4 f^{(3)}(\mu_i; y(\mu_i))}{3!} \int_0^1 s(s+1)(s+2)\,ds$$

$$= (\;)\int_0^1 (s^3 + 3s^2 + 2s)\,ds = \tag{7.1}$$

$$h^4 f^{(3)}(\mu_i, y(\mu_i))\, \frac{9}{4}\,\frac{1}{6} = \frac{3}{8}\, \underbrace{y^{(4)}(\mu_i)}_{\le M}\, h^4 \xrightarrow[h \to 0]{} 0$$

$O(h^4)$.

Then, from (3.1).

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} y'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$$

and

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt = \int_{t_i}^{t_{i+1}} P_2(t)\,dt + \int_{t_i}^{t_{i+1}} \frac{f^{(3)}(\xi(t), y(\xi(t)))}{3!}(t-t_i)(t-t_{i-1})(t-t_{i-2})\,dt$$

$$= \frac{h}{12}\left[23 f(t_i, y(t_i)) - 16 f(t_{i-1}, y(t_{i-1})) + 5 f(t_{i-2}, y(t_{i-2}))\right] + \frac{3}{8} y^{(4)}(\mu_i) h^4. \quad (7.2)$$

therefore,

$$\boxed{y_{i+1} = y_i + \frac{h}{12}\left[23 f(t_i, y(t_i)) - 16 f(t_{i-1}, y(t_{i-1})) + 5 f(t_{i-2}, y(t_{i-2}))\right] + \frac{3}{8} y^{(4)}(\mu_i) h^4} \quad (7.3)$$

Remark: $P_2(t)$ is a second order polynomial interpolating the points: (t_i, f_i), (t_{i-1}, f_{i-1}), (t_{i-2}, f_{i-2}).
and

$$\boxed{f(t, y(t)) = P_2(t) + \frac{f^{(3)}(\xi(t), y(\xi(t)))}{3!}(t-t_i)(t-t_{i-1})(t-t_{i-2})}$$

Summarizing,

The A-B 3 steps is given by

$$\begin{cases} w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2 \\ w_{i+1} = w_i + \frac{h}{12}\Big[23 f(t_i, w_i) - 16 f(t_i, w_{i-1}) \\ \qquad\qquad + 5 f(t_{i-2}, w_{i-2})\Big] \end{cases} \tag{7.4}$$

$i = 2, 3, \dots N-1.$

By substituting the exact solution into (7.4) and using (7.3)

$$y_{i+1} = y_i + \frac{h}{12}\Big[23 f(t_i, y_i) - 16 f(t_i, y_{i-1}) + 5 f(t_{i-2}, y_{i-2})\Big] + \frac{3}{8} y^{(4)}(\mu_i)\, h^4$$

<u>Def</u>: The difference obtained after substituting the exact solution into the numerical method (7.4) divided by h is called <u>the local truncation error for A-B three steps.</u>

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \frac{1}{12}\Big[23 f_i - 16 f_{i-1} + 5 f_{i-2}\Big] = \frac{3}{8} y^{(4)}(\mu_i)\, h^3$$

Show the Various explicit and Implicit Methods: Adams-Bashfort and Adams-Moulton from <u>Book</u>

Predictor-Corrector Method.

Predictor: A-B 4th order: (4 steps)

$$\begin{cases} w_0 = \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,\ w_3 = \alpha_3 \\ w_{i+1} = w_i + \frac{h}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right] \\ i = 3, 4 \dots N-1. \end{cases}$$

Corrector: A-M 3rd order (3 steps)

$$\begin{cases} w_0 = \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2 \\ w_{i+1} = w_i + \frac{h}{24}\left[9 f_{i+1} + 19 f_i - 5 f_{i-1} + f_{i-2}\right] \\ i = 2, 3, \dots N-1 \end{cases}$$

Procedure: i) R-K 4th order to obtain w_1, w_2, w_3.

ii) Use A-B 4th order to obtain w_{4p} (PREDICTOR)

$$\tilde{w}_{i+1} = w_i + \frac{h}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right]$$

iii) Use A-M 3rd order to obtain w_{i+1} (CORRECTOR)

$$w_{i+1} = w_i + \frac{h}{24}\left[9 f(t_i, \tilde{w}_{i+1}) + 19 f(t_i, w_i) - 5 f_{i-1} + f_{i-2}\right]$$

$\longrightarrow$ Show EXAMPLE IN BOOK!